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LOCATION OF A NEW FACILITY ON A LINEAR MARKET IN THE PRESENCE OF WEIGHTS

by

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ABSTRACT

In this paper, we study the trading areas of n facilities with unequal weights located on a finite linear market. Customers are uniformly distributed on the market. Their attraction to a facility is directly proportional to the weight of facility and inversely proportional to the distance separating them from the facility. The problem considered is that of determining which location will maximize the trading area of a new facility entering the market. A procedure which solves this problem in $O(n^2)$ time is presented and illustrated by means of a simple example.

Key words : location, market models, trading areas.

RÉSUMÉ

On étudie dans cet article les aires de marchés de n magasins de pouvoirs d'attraction (ou poids) inégaux et répartis sur un marché linéaire. Les clients sont distribués de façon uniforme sur le marché. Leur attraction à un magasin est directement proportionnelle au poids du magasin et inversement proportionnelle à la distance qui les sépare du magasin. Le problème consiste à déterminer la localisation d'un nouveau magasin s'implantant dans le marché afin de maximiser son aire de marché. On présente une procédure qui résout le problème en $O(n^2)$ temps. Cette procédure est illustrée à l'aide d'un exemple simple.

Mots-clés : localisation, modèles de marché, aires de marché.

1. Introduction

Ever since Hotelling [6] described the problem of the two ice-cream vendors on a beach, the subject of competitive location has attracted growing interest. The original Hotelling paper considers a so-called linear market in which customers are uniformly distributed and patronize the closest retailer. Hotelling showed that it is then optimal for the two vendors to locate themselves at the center of the beach: they each capture half of the market and moreover, this solution is stable. Various researchers (see, for example, [3], [4], and [8]) have since studied the entry and optimal locations of new competitors in an existing market. In particular, Hakimi [4] and ReVelle [8] sought to determine the sites of m new facilities on a network when n are already in place. Hakimi showed that the problem is NP-hard and ReVelle provided an algorithm which solves the problem optimally in "reasonable time".

In this paper, we consider the optimal location of one new facility on a linear market when n facilities are already located. However, unlike previous researchers, we no longer assume customers are attracted to the closest facility. Instead, we define an attraction function based on distance and facility weight.

2. Attraction Functions

The idea of defining attraction as a function of distance and weight has long been used in gravitational location models (see, for example, Reilly [7] and Hlavac and Little [5]). In order to formalize the models, let $\phi(x,i)$ denote the degree of attraction of customer x to facility i, d_{xi} the distance from x to i, and w_i, the weight of facility i. The weight is often assimilated with the size of the facility. In our interpretation however, it includes a whole range of factors which define the attractiveness of a facility such as size, service, parking space, price discounts,

etc. The attraction function $\phi(x,i)$ increases with respect to w_i and decreases with respect to d_{xi}. The definition commonly used in gravitational models is $\phi(x,i) = w_i/d_{xi}^2$. In this paper, we define the attraction function more simply as in Eiselt and Pederzoli [2] and Eiselt and Laporte [1], i.e. as $\phi(x,i) = w_i/d_{xi}$.

3. Trading Areas

The trading area of a facility is defined as the set of all points (customers) more attracted to that facility than to any other facility.

Consider first the case of facilities located in the plane. If they have the same weight, the attraction function $\phi(x,i) = w_i/d_{xi}$ defines a partition of the plane into Voronoi areas (see Rogers [9]) corresponding to the trading areas. When facilities have unequal weights, discontinuities occur: a customer located further from i than from j may patronize i. If $w_i > w_j$ and if d_{xi} denotes the Euclidean distance between x and i, the trading area of w_j is then a circle including j (but not centerd at j), while i captures the remainder of the plane. The general case of n facilities with non-equal weights is considerably more difficult to describe.

When n facilites are located on a line segment, their trading areas can be represented by a linear Voronoi diagram such as the one depicted in figure 1. In this example, there are four facilities P_1, P_2, P_3, P_4 on [0,100]. They have respective weights of 8, 4, 2, 1 and locations 50, 30, 90, 20. Following the procedure described in [1], the trading areas of the facilities are readily determined in $O(n^2)$ time, by considering in turn the facilities in increasing order of weights. A by-product of this procedure is: the trading area of the facility with the k^{th} smallest weight consists of no more than k unconnected pieces.

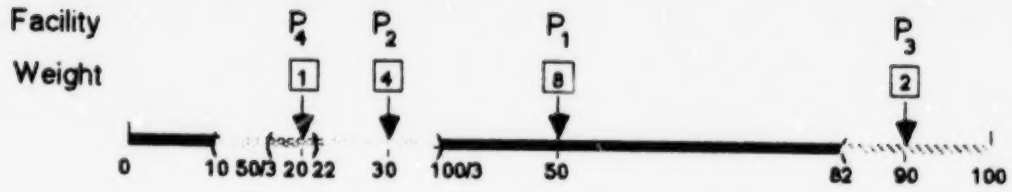

Figure 1: Trading Areas of Four Facilities

The boundaries between the various pieces are referred to as Voronoi points. Their maximal number is 2n.

4. Location of a New Facility on a Linear Market

Suppose now that n facilities are already located on a linear market [0,T]. If a new facility P_{n+1} enters the market, which location will maximize its trading area? This question is answered by considering the area facility P_{n+1} takes from each individual facility if it is located at a point $x \in [0,T]$. Define the functions:

$s_i(x)$: the area P_{n+1} captures from P_i if it is located at x;

and

$S(x) = \sum_{i=1}^{n} s_i(x)$: the total trading area of P_{n+1} if it is located at x.

We seek the value x* which maximizes S(x).

Each $s_i(x)$ is piecewise linear, as can be seen from the following example. Consider the interval and the four facilities given in section 3. Suppose the weight of the newly introduced facility P_5 is equal to 6. We will show how to compute $s_1(x)$ for $x \in [0,100]$. If P_5 is located at 0, the area it captures from P_1 is determined by the only point x_1 of equal attraction to these two facilities and belonging to [0,100]. This point is such that

$$6/x_1 = 8/(50-x_1), \text{ i.e. } x_1 = 150/7.$$

Therefore, P_5 captures from P_1 the whole of the interval [0,10]. As the location x of P_5 moves right, so does x_1. But the area captured from P_1 remains the same as long as $x<80/3$. Hence,

$$s_1(x)=10 \quad \text{for} \quad 0<x<80/3. \tag{1}$$

When x exceeds 80/3, P_5 starts attracting customers in the interval $[110/3, x_1]$ where $x_1 = (150+4x)/7$. Then

$$s_1(x) = (12x-110)/21 \quad \text{for} \quad 80/3<x<75/2. \tag{2}$$

When $x>75/2$, a new point of equal attraction to P_1 and P_5 appears left of P_5 so that the size of the area captured from P_1 is now equal to $(10-x_2)+(x_1-110/3)$ and

$$s_1(x) = (3040-72x)/21 \quad \text{for} \quad 75/2<x<40. \tag{3}$$

For $x>40$, P_5 no longer takes any area left of 10 and

$$s_1(x) = (12x-320)/21 \quad \text{for} \quad 40<x<140/3. \tag{4}$$

When $x=140/3$, x_2 becomes equal to 110/3 and the trading area of P_5 in the former territory of P_1 is $[x_2,x_1]$ where $x_2>110/3$ and $x_1<50$. The size of this area diminishes as x tends to 50 and eventually vanishes completely when x=50:

$$s_1(x) = (1200-24x)/7 \quad \text{for} \quad 140/3<x<50. \tag{5}$$

From this point, P_5 takes from P_1 the interval $[x_2,x_1]$ where $x_2>50$ and $x_1<82$:

$$s_1(x) = (24x-1200)/7 \quad \text{for} \quad 50<x<58. \tag{6}$$

Finally, x_1 becomes larger than 82 and

$$s_1(x) = (424-4x)/7 \quad \text{for} \quad 58<x<100. \tag{7}$$

The computation of $s_2(x)$, $s_3(x)$, and $s_4(x)$ follows a similar pattern. The four functions $s_i(x)$ (i=1,2,3,4) and $S(x)$ are represented in figure 2. Each s_i has breakpoints which are determined by the discontinuities in the trading area of P_i. For a given i, the number of these breakpoints is therefore $O(n)$. Since $S(x)$ is the sum of n piecewise linear functions, it has $O(n^2)$ breakpoints. $S(x)$ attains a maximum at one of them. We therefore suggest the following $O(n^2)$ procedure for determining x^* and $S(x^*)$: for i=1,...,n, compute $s_i(x)$ for all $x\varepsilon[0,T]$ and identify the set E_i of values of x at which the breakpoints of $s_i(x)$ occur; then evaluate $S(x)$ for all $x \varepsilon \bigcup_{i=1}^{n} E_i$. In the example, $x^*=30$ and the value of $S(x^*)$ is 35.143.

5. Conclusion

We have considered facilities of different weights on a linear market and an attraction function which is directly proportional to weight and inversely proportional to distance. The trading areas of the facilities are made up of disconnected intervals on the line segment. When n facilities are already located on the market, the location which maximizes the trading area of a new facility can be determined in $O(n^2)$ time.

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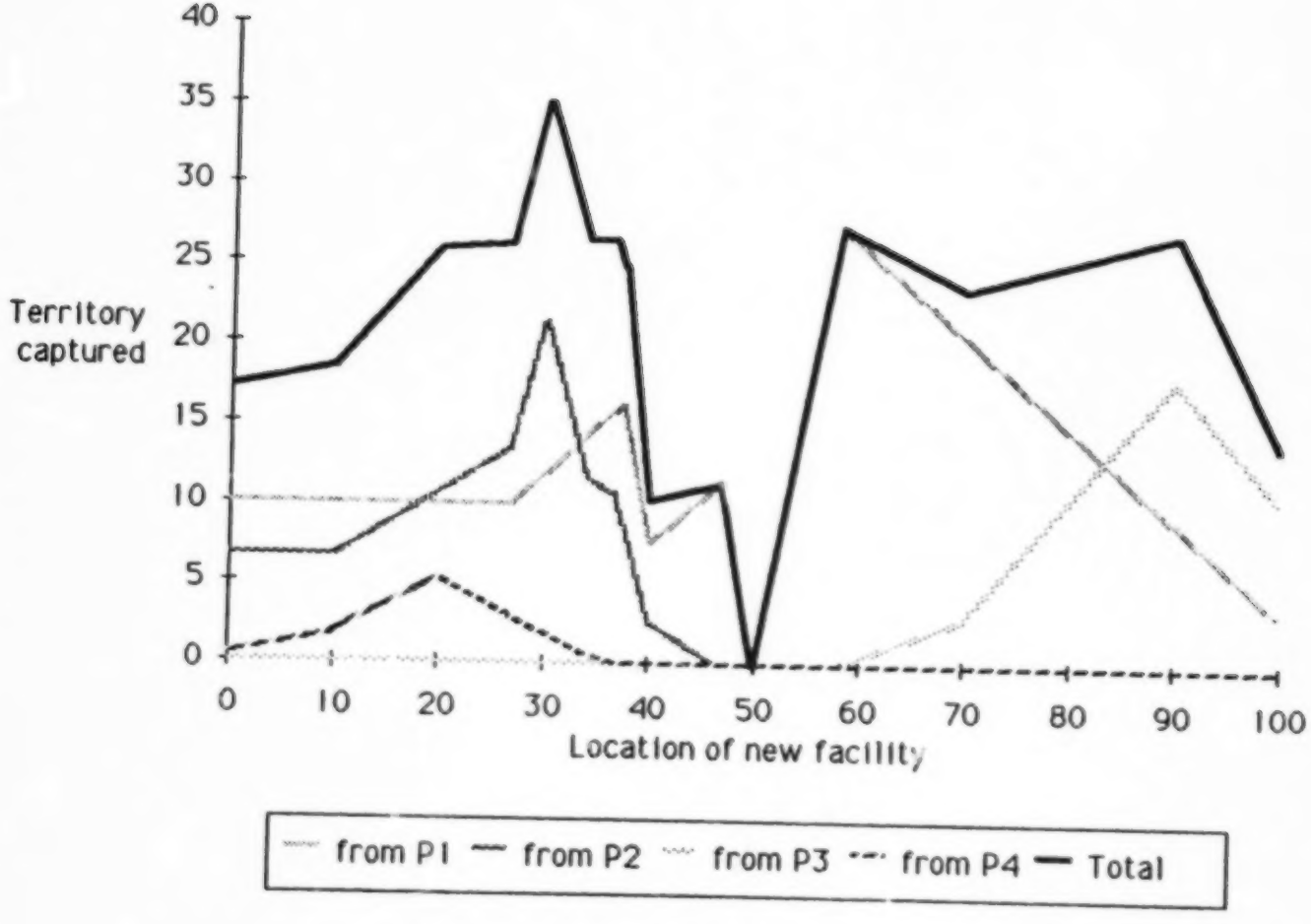

Figure 2: Territory captured by a new facility of weight 6

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